Metrics for scheduling problems with many machines

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Many algorithms exist to solve scheduling problems

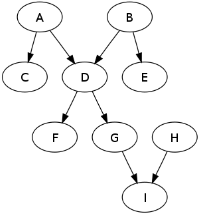
Algorithm	Exact	Approximation
Advantage	The objective function is calculated without any error	Good speed and relative simplicity
Disadvantages	Time-consuming calculations	There are no estimates of the objective function error

Approximate polynomial scheme

- Guaranteed polynomial complexity
- Evaluation of the solutions accuracy: the accuracy value forms the complexity of the algorithm

Jobs $j \in N = \{1, \ldots, n\}$ are serviced on machines $i \in M = \{1, \ldots, m\}$. Interrupts are not allowed. The machine serves one job at a time.

- release date r_j,
- due date d_j,
- processing times $0 \le p_{ij} \le +\infty$ on machine $i \in M$.



Relations between jobs are given by the graph G. Splitting N jobs into subsets of N_i jobs generates a schedule.

For each set N_i you need to find a sequence of orders π_i for the machine i.

$Pred(j)$ - the set of jobs served before j according to the graph G, $(k \to j)_{\pi_i}$ jobs processed on i before j in the π_i.

A starting time s_j for all $j \in N_i, i = 1, \ldots, m$.

The starting time of a job $j \in N_i$, $i = 1, \ldots, m$ in the schedule π:

$$s_j(\pi) = \max \left\{ r_j, \max_{k \in Pred(j)} (s_k(\pi) + p_{ik}), \max_{(k \to j)_{\pi_i}} (s_k(\pi_i) + p_{ik}) \right\}. \qquad (1)$$

The compliting time of a job $j \in N_i$ in the schedule π:

$$C_j(\pi) = s_j(\pi) + p_{ij}, \, j \in N_i.$$

The schedule π is called *feasible*, if $r_j \leq s_j(\pi)$ and $C_j(\pi) \leq s_k(\pi)$ for all arcs $(j,k) \in G$.

Remark

If a schedule π is known, the starting times S can be uniquely determined and vice versa, if all starting times S (together with the sets $N_1, \ldots, N_m$) are known, this uniquely identifies the resulting schedule π.

The optimization criterion is to minimize the maximum lateness:

$$L_{\max} = \min_{\pi} \max_{j \in N} \{C_j(\pi) - d_j\}.$$

If $d_j = 0$ for all jobs $j \in N$, the objective turns into the makespan criterion.

$$C_{\max} = \min_{\pi} \max_{j \in N} \{C_j(\pi)\}.$$

An instance of A will be called **some** NP - **difficult subproblem** of the $P|prec, r_j|L_{\max}$.
Many investigated NP - hard problems of the form $P|prec, r_j|L_{\max}$, can be considered as an instance of A, in particular:

- $P|intree, r_j, p_j = 1|C_{\max}$ [Brucker (1977)];
- $P|outtree, p_j = 1|L_{\max}$ [Brucker (1977)];
- $P2|chains|C_{\max}$ [Du (1991)];
- $P||C_{\max}$ [Garey(1978)];
- $P2||C_{\max}$ [Lenstra (1977)];
- $P|prec, p_j = 1|C_{\max}$ [Ullman (1975)].

Definition

A metric for A and B is a function that satisfies the properties:

$$\rho(A,B) = 0 \Leftrightarrow A = B \qquad (2)$$

$$\rho(A,B) = \rho(B,A) \qquad (3)$$

$$\rho(A,C) \leq \rho(A,B) + \rho(B,C) \qquad (4)$$

for all A, B, C.

For two arbitrary instances A and B of the problem$\{P, Q, R\} \mid prec, r_j \mid L_{\max}$ we define the following functions:

$$\begin{cases} \rho_d(A,B) = \max_{j \in N}\{d_j^A - d_j^B\} - \min_{j \in N}\{d_j^A - d_j^B\}; \\ \rho_r(A,B) = \max_{j \in N}\{r_j^A - r_j^B\} - \min_{j \in N}\{r_j^A - r_j^B\}; \\ \rho_p(A,B) = \sum_{j \in N}\left(\max_{i \in M}(p_{ij}^A - p_{ij}^B)_+ + \max_{i \in M}(p_{ij}^A - p_{ij}^B)_-\right); \\ \rho(A,B) = \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B), \end{cases} \qquad (5)$$

Under the metric $rho(A,B)$, $P|prec, r_j|L_{\max}$ we will understand a function that satisfies the properties (2-5)

Definition

Let A be an instance with the set of jobs N and the precedence relation G. We say that instance B with the same set of jobs inherits the parameter x from the instance A if $x_j^B = x_j^A$ for all $j \in N$.

Let the instance D inherit all parameters from the instance A except the values $\{d_j, r_j, p_{ij} \mid j \in N, i \in M\}$, and let $\tilde{\pi}^D$ be an approximate solution of the instance D satisfying the condition

$$L^B_{\max}(\tilde{\pi}^B) - L^B_{\max}(\pi^B) \le \delta_B. \tag{6}$$

Then

$$0 \le L^A_{\max}(\tilde{\pi}^B) - L^A_{\max}(\pi^A) \le \rho(A, B) + \delta_B. \tag{7}$$

Definition

Let $\mathfrak{A}$ be the space, where each point represents the data of an instance of the problem $P \mid prec, r_j \mid L_{\max}$. The sub-space $\tilde{\mathfrak{A}} \subset \mathfrak{A}$ is called **P-cone**, if all instances represented by points of this sub-space can be solved by a polynomial or pseudo-polynomial algorithm. These points in $\tilde{\mathfrak{A}}$ are called **P-points**.

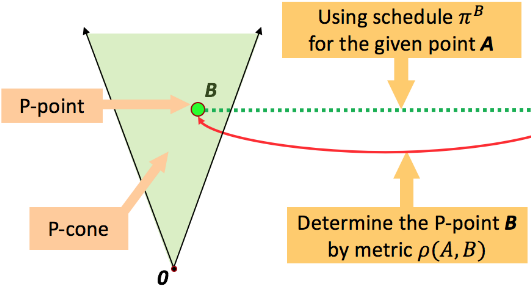
Definition

Let there be a point (instance) $A \notin \tilde{\mathfrak{A}}$. Using some metric ρ, we can construct a projection onto the space $\tilde{\mathfrak{A}}$ with respect to A. The resulting point (instance) $B \in \tilde{\mathfrak{A}}$ is called the projection of A by the metric ρ.

Definition

The sub-space $\tilde{\mathfrak{A}}^{\epsilon}_{\rho}(A) \in \tilde{\mathfrak{A}}$ is called an ϵ–projection of A by the metric ρ if for each of its points $x \in \tilde{\mathfrak{A}}$, the following inequality is satisfied:

$$L^A_{max}(\pi^x) - L^A_{max}(\pi^A) \leq \epsilon.$$



- We change the parameters $\{(r_j^A, p_j^A, d_j^A) | j \in N\}$ of the original instance $A = \{G, (r_j^A, p_j^A, d_j^A)\}$, where $j \in N, A \notin \tilde{\mathfrak{A}}$, so that the projection of A by the metric ρ gives an instance $B = \{G, (r_j^B, p_j^B, d_j^B) | j \in N\}$ in the P-cone.
- We find an optimal schedule π^B for the instance B. According to Theorem 7, we apply the schedule π^B to the initial instance A. As a result, we obtain the following estimate of the absolute error:

$$0 \leq L^A_{\max}(\pi^B) - L^A_{\max}(\pi^A) \leq \rho(A, B).$$

We consider the P-cone when the parameters of the jobs satisfy the following k linearly independent inequalities:

$$XR + YP + ZD \leq H, \tag{8}$$

where $R = (r_1, \ldots, r_n)^T$, $P = (p_1, \ldots, p_n)^T$ ($p_j \geq 0$ for all $j \in N$), $D = (d_1, \ldots, d_n)^T$ and X, Y, Z are matrices of dimension $k \times n$, $H = (h_1, \ldots, h_k)^T$ is a k-dimensional vector (the upper index ^{T} denotes the transpose operation). Then in the class of instances (8), we determine an instance B with minimal distance $\rho(A, B)$ to the original instance A by solving the following problem:

$$\begin{cases} (x^d - y^d + x^r - y^r) + \sum\limits_{j \in N} x_j^p \longrightarrow \min \\ y^d \leq d_j^A - d_j^B \leq x^d \quad \text{for all } j \in N, \\ y^r \leq r_j^A - r_j^B \leq x^r \quad \text{for all } j \in N, \\ -x_j^p \leq p_j^A - p_j^B \leq x_j^p \quad \text{for all } j \in N, \\ 0 \leq x_j^p \quad \text{for all } j \in N, \\ XR^B + YP^B + ZD^B \leq H. \end{cases} \tag{9}$$

The linear programming problem (9) with $3n + 4 + n$ variables $(r_j^B, p_j^B, d_j^B,\ x_d,\ y_d,\ x_r,\ y_r$, and x_j^p, $j = 1, \dots, n)$ and $7n + k$ inequalities can sometimes be solved with a polynomial number (in n and k) of operations, given the specificity of the constraints of the problem (9). For problem $1|r_j|L_{\max}$, there are two types of non-trivial P-points [Hoogeveen (1996)]:

$$\begin{cases} d_1 \leq \ldots \leq d_n, \\ d_1 - r_1 - p_1 \geq \ldots \geq d_n - r_n - p_n, \end{cases} \tag{10}$$

An optimal solution of problem $1|r_j|L_{\max}$ can be found in $O(n^3 \log n)$ operations. The linear programming problem (9) can be solved in $O(n \log n)$ operations. The minimum absolute error of the maximum lateness can be found in polynomial time, in this case with $O(n)$ operations.

$$\max_{k \in N}\{d_k - r_k - p_k\} \leq d_j - r_j \quad \text{for all} \quad j \in N. \tag{11}$$

An optimal schedule can be found in $O(n^2 \log n)$ operations.

New types of P-point were found [Lazarev (2019)]

- $$r_i \le r_j \Rightarrow d_i \ge d_j \quad \text{for all } i, j \in N;$$

  $$d_j - p_j \le d_{\min}(N) \quad \text{for all } j \in N,$$

  algorithm of solution with complexity $O(n^2)$ operations

- $$r_i \le r_j \Rightarrow d_i - p_i \ge d_j \quad \text{for all } i, j \in N, \quad i \ne j.$$

  solution algorithm with complexity $O(n \log n)$ operations

Assume that the instance B inherits all parameters from the instance A except the due dates $\{d_j \mid j \in N\}$, and let $\tilde{\pi}^B$ be an approximate solution for the instance B satisfying the condition

$$0 \le L^B_{\max}(\tilde{\pi}^B) - L^B_{\max}(\pi^B) \le \delta_B, \qquad (12)$$

where π^B is an optimal solution, i.e., it satisfies the condition

$$L^B_{\max}(\pi^B) \le L^B_{\max}(\pi) \quad \text{for all} \quad \pi. \qquad (13)$$

Then we obtain

$$0 \le L^A_{\max}(\tilde{\pi}^B) - L^A_{\max}(\pi^A) \le \rho_d(A, B) + \delta_B.$$

Let there be some instance A of a problem $\alpha^A|\beta^A|L_{\max}$ belonging to the class of NP-hard problems and a known approximate schedule $\tilde{\pi}^B$ (or even an optimal one π^B) for the instance B for the problem $\alpha^A|\beta^A|C_{\max}$ with an absolute error not exceeding $\delta_B \ge 0$. In the instance B, we have $d_j^B = 0$ for all $j \in N$ and thus, from Lemma 13, we obtain the following bound:

$$0 \le L^A_{\max}(\tilde{\pi}^B) - L^A_{\max}(\pi^A) \le \rho_d(A, B) + \delta_B = \max_{j \in N}\{d_j^A\} - \min_{j \in N}\{d_j^A\} + \delta_B.$$

In fact, the obtained estimate allows to estimate the transition from the objective function $L_{\max}$ to the makespan $C_{\max}$.

Conclusions:

- For the first time introduced **metrics** in the scheduling with which we can build approximate polynomial algorithms and obtain **absolute error estimate** of the objective function.
- In fact, the best use of previously found **polynomial solvable sub-cases** of the studied problem occurs.
- With this approach, it is possible to quantify the textbfmeasure of polynomial unsolvability of the problem.

Plans:

- comparison of the metric approach with other approaches (B&B, dynamic programming, etc.).)
- the use of a metric algorithm to other problems of discrete optimization

Thank you for your attention!