Composite optimization for the resource allocation problem

Anastasiya Ivanova

Moscow Institute of Physics and Technology

anastasiya.s.ivanova@phystech.edu

joint work with P. Dvurechensky and A. Gasnikov

IDP 2018

Problem statement. Primal problem.

There is a Center and n producers which produce one product.
$f_k(x_k)$, $k = 1, \dots, n$ – cost functions;
$x_k \in \mathbb{R}$ – the volume of product produced by the producer k in one year;
y_k – the volume of product which is purchased from the producer k;
C – the lower bound for the total production volume per year by all producers.

Problem statement. Primal problem.

There is a Center and n producers which produce one product.
$f_k(x_k)$, $k = 1, \ldots, n$ – cost functions;
$x_k \in \mathbb{R}$ – the volume of product produced by the producer k in one year;
y_k – the volume of product which is purchased from the producer k;
C – the lower bound for the total production volume per year by all producers.

$$(P) \qquad f(\mathrm{x}) = \sum_{k=1}^{n} f_k(x_k) \to \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\ x_k \geqslant y_k; \\ y_k \geqslant 0,\ x_k \geqslant 0,\ k=1,\ldots,n,}}$$

where cost functions $f_k(x_k)$ $k = 1, \ldots, n$ are increasing and μ-strongly convex.

Problem statement. Dual problem.

Introducing dual variables p_k, $k = 1, ..., n$ and using the duality theory, we obtain the dual problem (up to a sign)

$$(D) \qquad \varphi(\mathrm{p}) = \sum_{k=1}^{n} \Big\{ p_k x_k(p_k) - f_k(x_k(p_k)) \Big\} - C \min_{k=1,\dots,n} p_k \to \min_{\mathrm{p} \geqslant 0}$$

where

$$x_k(p_k) = \operatorname*{argmax}_{x_k \geqslant 0} \Big\{ p_k x_k - f_k(x_k) \Big\}, \quad k = 1,\, 2,\, \dots,\, n.$$

Subgradient method for the resource allocation

Input: $\varepsilon > 0$ – accuracy, p^0 – starting point.

1. Set the stepsize $h = \frac{\varepsilon}{nC^2}$.
2. Given the price vector p^t for the current year, producers calculate the optimal production plan for these prices as

$$x_k(p_k^t) = \underset{x_k \geqslant 0}{\operatorname{argmax}} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$

   and communicates this information to the Center.
3. The Center determines the shares of purchases for each producer, i.e. forms a vector $\lambda(\mathrm{p}^t)$ as $\lambda^t = (\lambda_1^t, \ldots, \lambda_n^t)^\top$, where $\sum_{k=1}^{n} \lambda_k^t = 1$, $\lambda_k^t \geqslant 0$ if $k \in \operatorname{Arg}\min_{j=1,\ldots,n} p_j$ and $\lambda_k^t = 0$, if $k \notin \operatorname{Arg}\min_{j=1,\ldots,n} p_j^t$ and sends this vector to all factories.
4. Each factory adjusts the price for the next year as follows

$$\mathrm{p}^{t+1} = \left(\mathrm{p}^t - h(\mathrm{x}(\mathrm{p}^t) - C\lambda(\mathrm{p}^t))\right)_+.$$

Convergence rate

To state the convergence rate result, we need introduce an upper bound for the optimal value of the prices.

Lemma 1

Let the p^* be a solution to the dual problem (D). Then

$$\|\mathrm{p}^*\|_2 \leqslant \sqrt{n} p_{max}.$$

where

$$p_{max} := \frac{n}{C} \left(\sum_{k=1}^{n} f_k\left(\frac{2C}{n}\right) - \sum_{k=1}^{n} f_k(0) \right). \quad (1)$$

Convergence rate

Theorem 1

Let Algorithm 1 be run with starting point p^0 satisfying $0 \leqslant p_k^0 \leqslant p_{max},\ k = 1, \dots, n$ for

$$N = \left\lceil \frac{164(Cnp_{max})^2}{\varepsilon^2} \right\rceil$$

steps. Then

$$f(\mathrm{x}^N) - f(\mathrm{x}^*) \leqslant \varepsilon,\ C - \sum_{k=1}^{n} x_k^N \leqslant \frac{\varepsilon}{3p_{max}}, \tag{2}$$

where $\mathrm{x}^N = \frac{1}{N}\sum_{t=0}^{N-1}\mathrm{x}(\mathrm{p}^t)$.

Composite gradient method for the resource allocation problem

The problem (D) can be rewritten as

$$\varphi(p_1, \ldots, p_n) = \psi(p_1, \ldots, p_n) + g(p_1, \ldots, p_n),$$

where

$$\psi(p_1, \ldots, p_n) = \sum_{k=1}^{n} \Big\{ p_k x_k(p_k) - f_k(x_k(p_k)) \Big\} = \langle \mathrm{p}, \mathrm{x}(\mathrm{p}) \rangle - f(\mathrm{x}(\mathrm{p}))$$

is convex function, which gradient satisfies Lipschitz condition

$$\left\| \nabla \psi(\mathrm{p}^1) - \nabla \psi(\mathrm{p}^2) \right\|_2 \leqslant L_\psi \left\| \mathrm{p}^1 - \mathrm{p}^2 \right\|_2, \ \forall\, \mathrm{p}^1, \mathrm{p}^2 \geqslant 0,$$

where $L_\psi = \dfrac{n}{\mu}$ and

$$g(p_1, \ldots, p_n) = -C \min_{k=1,\ldots,n} p_k$$

is convex non smooth function.

General composite projected gradient method

Input: $N > 0$ – number of steps, L_ψ – Lipschitz constant of gradient ψ, p^0 – starting point.

1. Find

$$x_k(p_k^t) = \operatorname*{argmax}_{x_k \geqslant 0} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$

2. Do the step

$$\mathrm{p}^{t+1} = \operatorname*{argmin}_{\mathrm{p} \geqslant 0} \left\{ \left\langle \nabla \psi(\mathrm{p}^t), \mathrm{p} - \mathrm{p}^t \right\rangle - C \min_{k=1,\ldots,n} p_k + \frac{L_\psi}{2} \left\| \mathrm{p} - \mathrm{p}^t \right\|_2^2 \right\}$$

Step of the composite gradient method

Lemma 2

Let $\tilde{\mathrm{p}}^{t+1} = \mathrm{p}^t - \frac{1}{L_\psi}\mathrm{x}(\mathrm{p}^t)$. Then p^{t+1} in (2) is defined as follows

- If $\sum_{k=1}^{n} \left(-\tilde{p}_k^{t+1}\right)_+ > \frac{C}{L_\psi}$ then $p_{center}^{t+1} = 0$ and

$$p_k^{t+1} = \max\left(0,\ \tilde{p}_k^{t+1}\right),\ \ k = 1, \dots, n.$$

- Else $p_{center}^{t+1} > 0$ is a solution of equation

$$\sum_{k=1}^{n} \left(p_{center}^{t+1} - \tilde{p}_k^{t+1}\right)_+ = \frac{C}{L_\psi}$$

and

$$p_k^{t+1} = \max\left(p_{center}^{t+1},\ \tilde{p}_k^{t+1}\right),\ \ k = 1, \dots, n.$$

Step of the composite gradient method

Note that the step also can be rewritten as

$$\mathrm{p}^{t+1} = \left[\mathrm{p}^t - \frac{1}{L_\psi}\left(\mathrm{x}(\mathrm{p}^t) - C\lambda(\mathrm{p}^{t+1})\right)\right]_+,$$

where $\lambda(\mathrm{p}^{t+1})$ is such that $\sum\limits_{k=1}^{n}\lambda_k(p_k^{t+1}) = 1,\ \lambda_k(p_k^{t+1}) \geqslant 0$ if $k \in \operatorname*{Arg\,min}\limits_{j=1,\dots,n} p_j^{t+1}$ and $\lambda_k(p_k^{t+1}) = 0$, if $k \notin \operatorname*{Arg\,min}\limits_{j=1,\dots,n} p_j^{t+1}$.
By Lemma 2, the solution of step can be written as

$$\mathrm{p}^{t+1} = \left[\tilde{\mathrm{p}}^{t+1} + \frac{C}{L_\psi}\lambda(\mathrm{p}^{t+1})\right]_+,$$

where

$$\lambda_k(p_k^{t+1}) = \frac{L_\psi}{C}\left(p_{center}^{t+1} - \tilde{p}_k^{t+1}\right)_+.$$

Input: $N > 0$ – number of steps, L_ψ – Lipschitz constant, p^0

1. Knowing the prices p_k^t producers calculate the optimal plan

$$x_k(p_k^t) = \operatorname*{argmax}_{x_k \geqslant 0} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$

2. The Center forms a prediction for the lowest possible prices

$$\tilde{p}_k^{t+1} = p_k^t - \frac{1}{L_\psi} x_k(p_k^t), \quad k = 1, 2, \ldots, n.$$

3. The Center determines the price p_{center}^{t+1}
   - If $\sum\limits_{k=1}^{n} \left(-\tilde{p}_k^{t+1}\right)_+ > \frac{C}{L_\psi}$ then $p_{center}^{t+1} = 0$;
   - Else $p_{center}^{t+1} > 0$ and solves equaltion

$$\sum_{k=1}^{n} \left(p_{center}^{t+1} - \tilde{p}_k^{t+1}\right)_+ = \frac{C}{L_\psi}$$

4. Each producer adjusts the price for the next year as follows

$$p_k^{t+1} = \max\left(p_{center}^{t+1}, \, \tilde{p}_k^{t+1}\right), \; k = 1, \ldots, n.$$

Convergence rate

Theorem 2

Let Algorithm 2 be run for N steps with starting point p^0 satisfying $0 \leqslant p_k^0 \leqslant p_{max},\ k = 1, \dots, n$, where p_{max} is given in (1). Then

$$f(\mathrm{x}^N) - f(\mathrm{x}^*) \leqslant f(\mathrm{x}^N) + \varphi(\mathrm{p}^*) \leqslant \varphi(\mathrm{p}^N) + f(\mathrm{x}^N) \leqslant \frac{82 p_{max}^2 n^2}{N\mu},$$

$$\left[C - \sum_{k=1}^{n} x_k^N\right]_+ \leqslant \frac{82 p_{max} n^2}{3N\mu}$$

where $\mathrm{p}^N = \frac{1}{N}\sum_{t=1}^{N} \mathrm{p}^t$ and $\mathrm{x}^N = \frac{1}{N}\sum_{t=0}^{N-1} \mathrm{x}(\mathrm{p}^t)$.

References

Gasnikov A. Universal gradient descent. arXiv preprint arXiv:1711.00394

Ivanova A., Dvurechensky P., Gasnikov A. Composite optimization for the resource allocation problem. arXiv preprint arXiv:1810.00595

Nesterov Yu. Primal-dual subgradient methods for convex problems. Math. Program. Ser. B. – 2009. – V. – 120(1). – P. 261–283.

Nesterov Yu., Shikhman V. Distributed Price Adjustment Based on Convex Analysis. J. Optimization Theory and Applications 172(2): 594-622 (2017)

Thank you!