# Algorithms of “cloud approximation” for nonconvex sets in a finite-dimensional space

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# Applications areas

- reachable set of controllable system
- trajectory of dynamical system
- estimation of the local extrema regions of the function
- multicriteria optimization
- data analysis
- …

# Problems

- phase estimation
- nonlocal optimal control methods
- search for a global function extremum
- variational inequalities
- impacts of normalization
- …

# “Cloud approximation” Term

- irregular grid
- randomized mesh
- stochastic grid
- set of points
- ...

# The optimal control problem
## with box constraints

$$\dot{x} = f(x, u, t)$$

$$x(t_0) = x^0, \quad t \in T = [t_0, t_1]$$

$$u(t) \in U = \{u \in R^r : \underline{u}_i \le u_i \le \overline{u}_i\}$$

$$I(u) = \varphi(x(t_1)) \to \min$$

# The approximation problem of the reachable set

## Stabilization of a nonlinear pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u_1 - \sin x_1$$

$$| u_1(t) | \leq 1$$

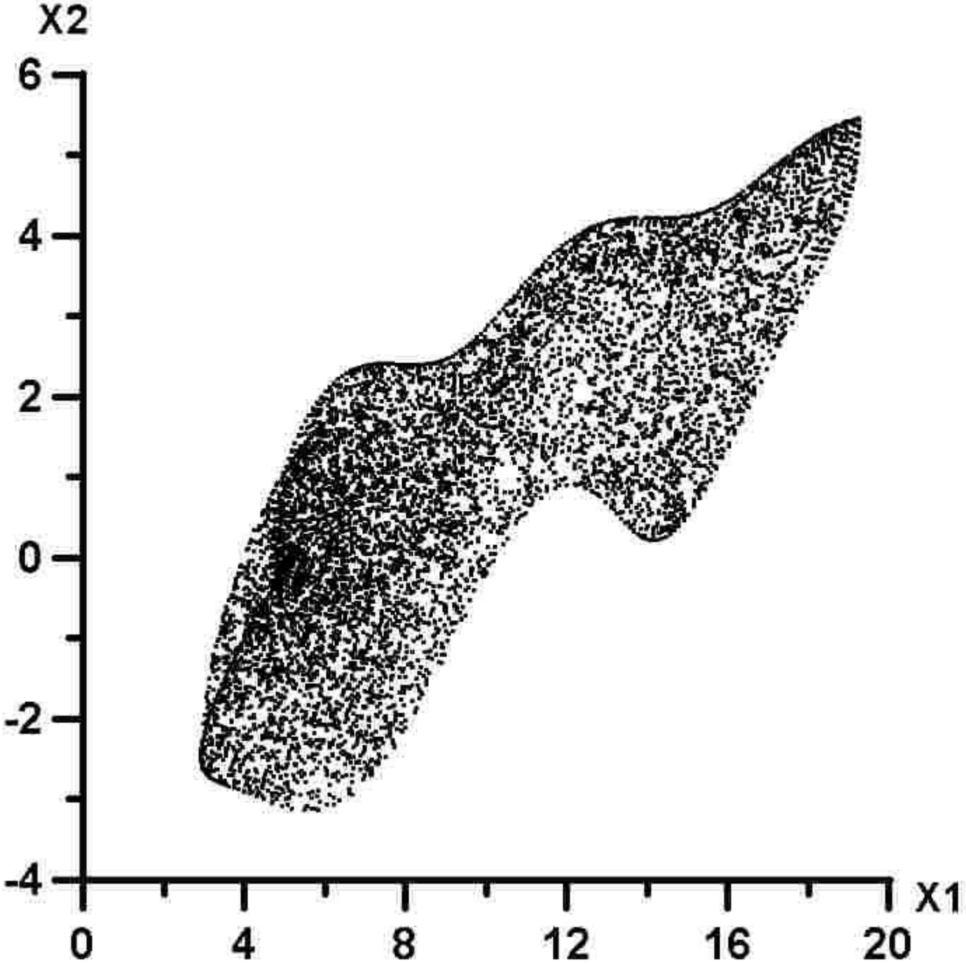
$$t \in [\, 0, 5]$$

$$x(0) = (\, 5, 0)$$

$$I(u) = x_1^2(5) + x_2^2(5) \to \min$$

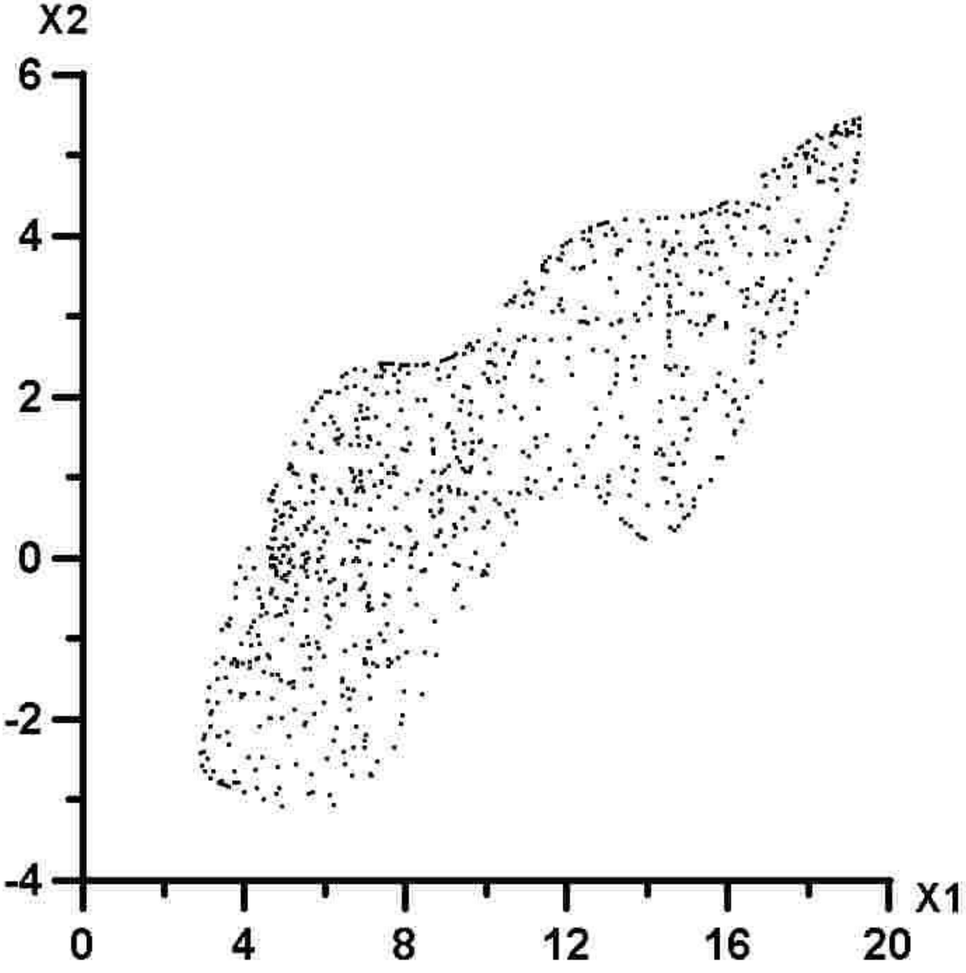
# The approximation problem of the reachable set

## Stabilization of a nonlinear pendulum



# The approximation problem of the reachable set

## Stabilization of a nonlinear pendulum



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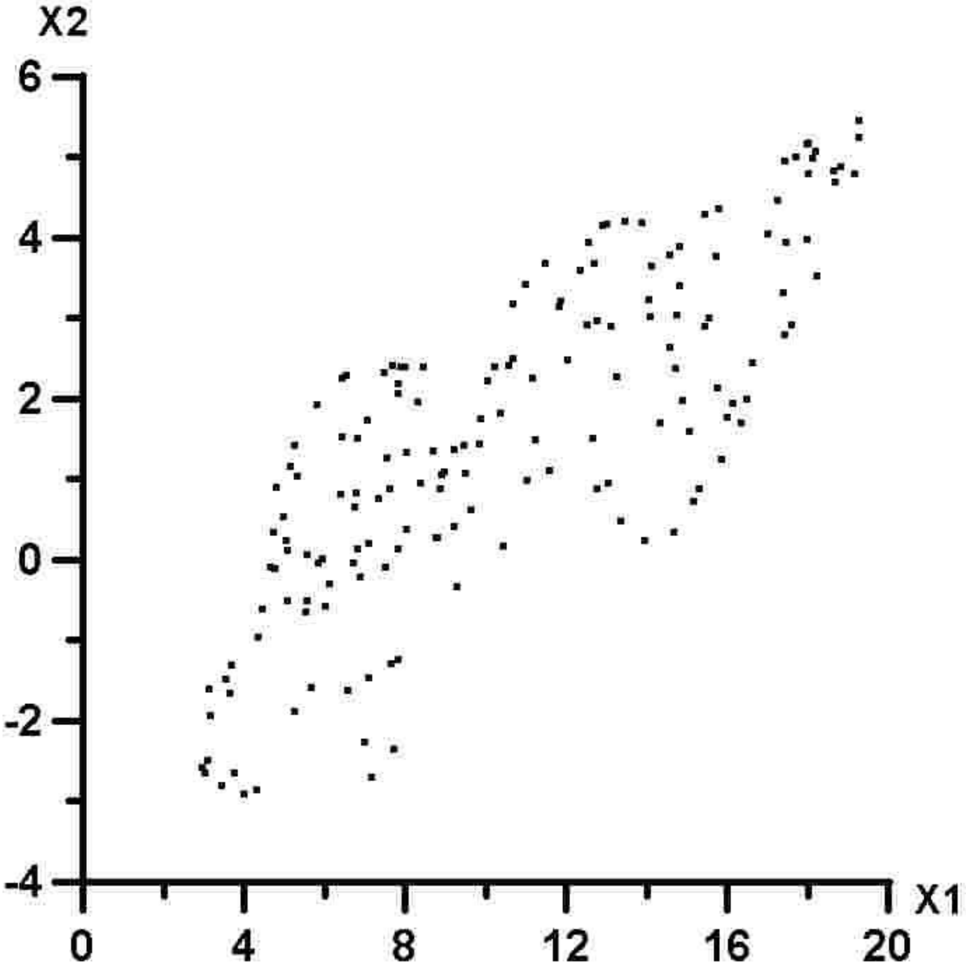

# The approximation problem of the reachable set

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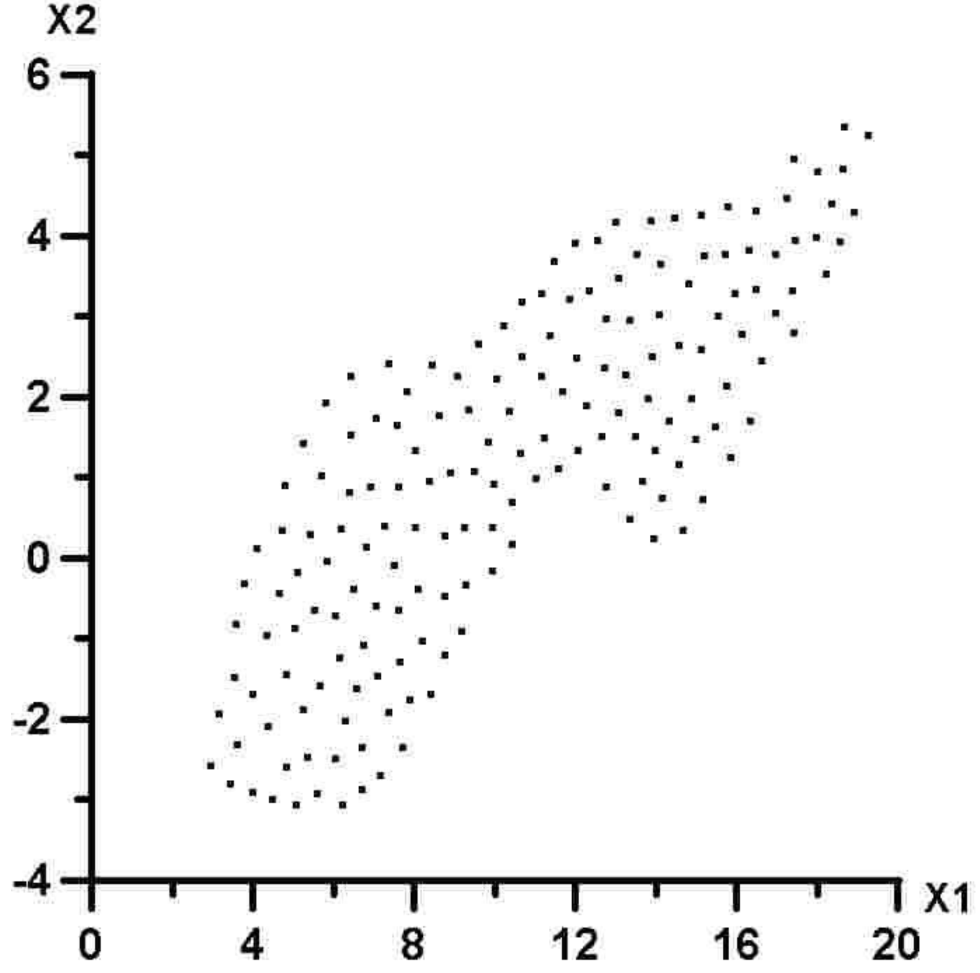


# A common approach
## input-output system

- uniform approximation of input parameters
- processing of multiple output parameters

# Approach to the construction of algorithms

- only scalable basic operations (Euclidean norms, $L_1$-norms, ...)
- no more than subquadratic computing schemes
- visualization
- …

# Test problems 01

$$\dot{x}_1 = x_2$$

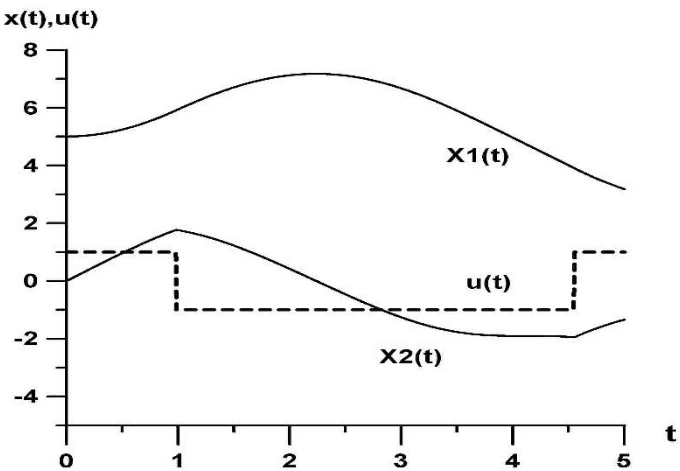
$$\dot{x}_2 = u_1 - \sin x_1$$

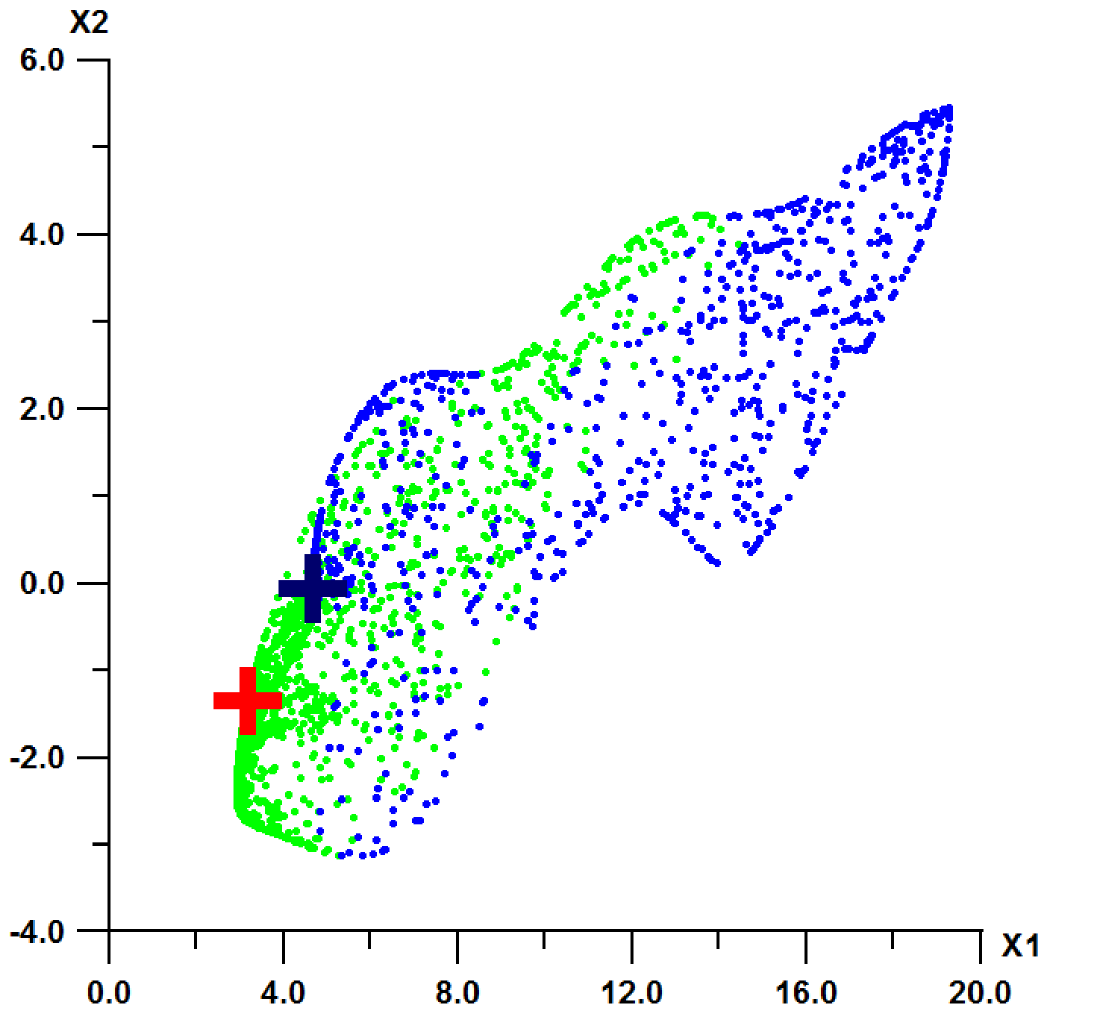
$$x(0) = (5, 0) \quad t \in [0, 5] \quad |u_1(t)| \le 1$$

$$I(u) = x_1^2(5) + x_2^2(5) \to \min$$

**Iteration – 1000,**
**CPU time is 7591 sec.,**
**Number of solved Cauchy problems – 186015**

| N | Functional | Value |
|---|---|---|
| 1 | 1.190817e+01 | 0.457 |
| 2 | 2.182900e+01 | 0.543 |



# The global optimization problem Approach

- Identification and evaluation of local extremum areas
- decomposition of the original problem into subtasks with the "small" reachable sets

# The global optimization problem formulation

$$f(x) -> \min, \; x \in X$$

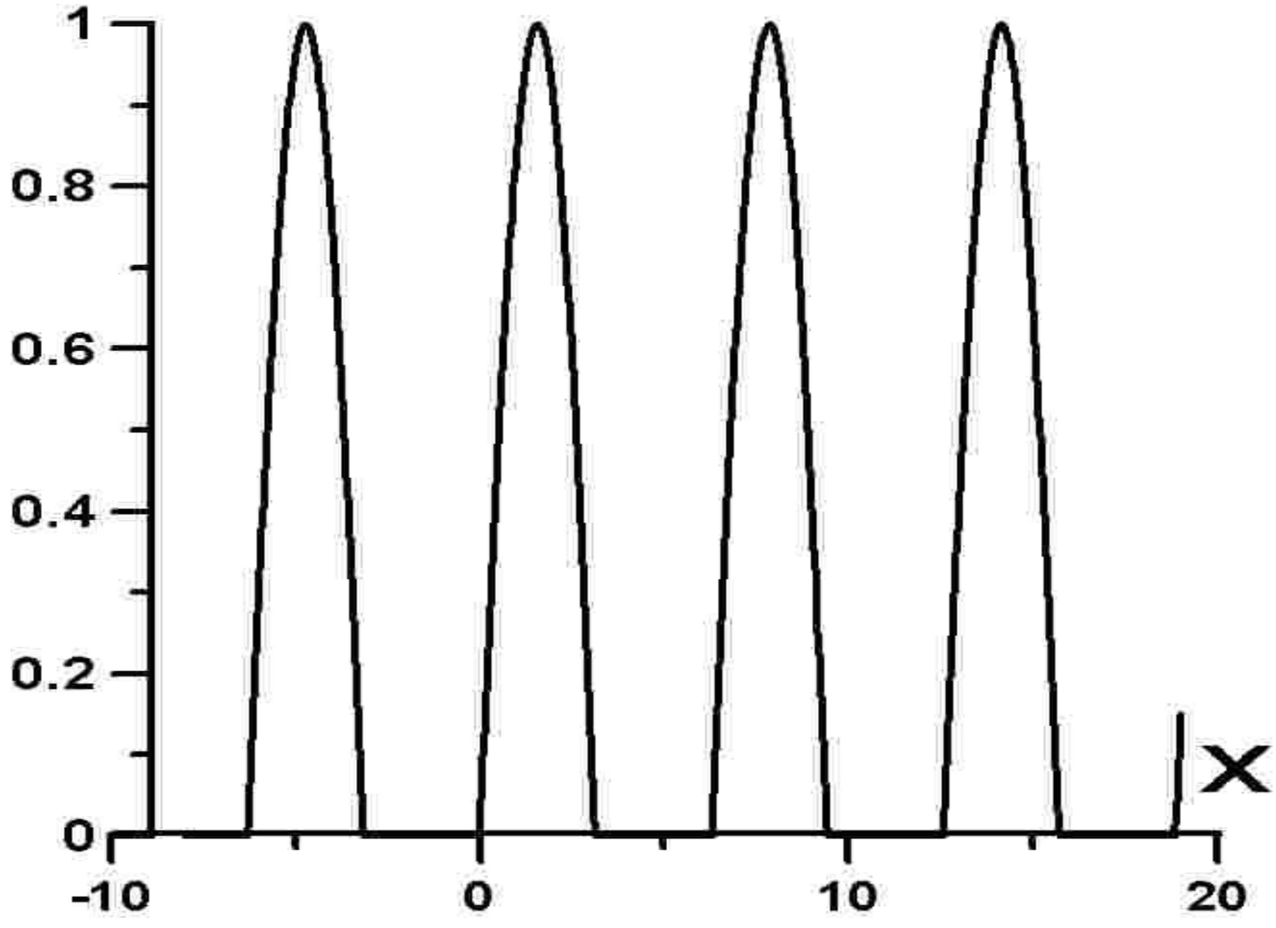
$$f(x) -> \min, \; x_l \leq x \leq x_g$$

$$x \in X = \{ x_l \leq x \leq x_g \} : f(x) = f^*$$

$$X^* \in X : \forall x \in X^* \; f(x) = f^*$$

# Example

**f(x)=max(sin(x),0)**



# Classification of optimization problems by the number of extrema

- “Low extremes” – 2-5 extrema
- “Medium-extreme” – 5-30 extrema
- “Multiextremal” – 30-"many extrema“
- with a multivalued solution – infinitely many extrema

# Example

## Optimization of Atomic Molecular Potentials
## Morse Potential

- the number of variables = 3 * number of atoms
- the number of local extrema grows as an exponent of the atoms number
- with the number of atoms = 147, the estimate of the number of local extrema is 10**60
- officially registered record is 240 atoms. The number of extrema > 10**100?

Cambridge Cluster Database

# Classification of optimization problems by structure of extrema

- several extrema with different values of the function
- several extrema with the same value of the function
- a set of solutions with the same value of the function
- sets of solutions with different values of the function

# Global optimization
## problems and hopes

- the problem of the volume ratio of the search set and the possibilities of searching, the resource of probes

- *the problem dimension is* 100,
*box* [0,1], *volume* 1

- *the problem dimension is* 100,
*box* [0,2], *volume* 2**100=(1024)**10 =
1.26*10**30

# Global optimization
## problems and hopes

- in order to solve the global optimization problem, one must be able to solve **only two** problems:

1) find one point in the region of the global extremum attraction;
2) find the minimum of unimodal function

# Open problems
## Control of trajectory beams

- Control in conditions of indeterminacy (in the system both control and perturbation)

- The problem of formulating the "nuclear problem“

- Example: impacts of normalization

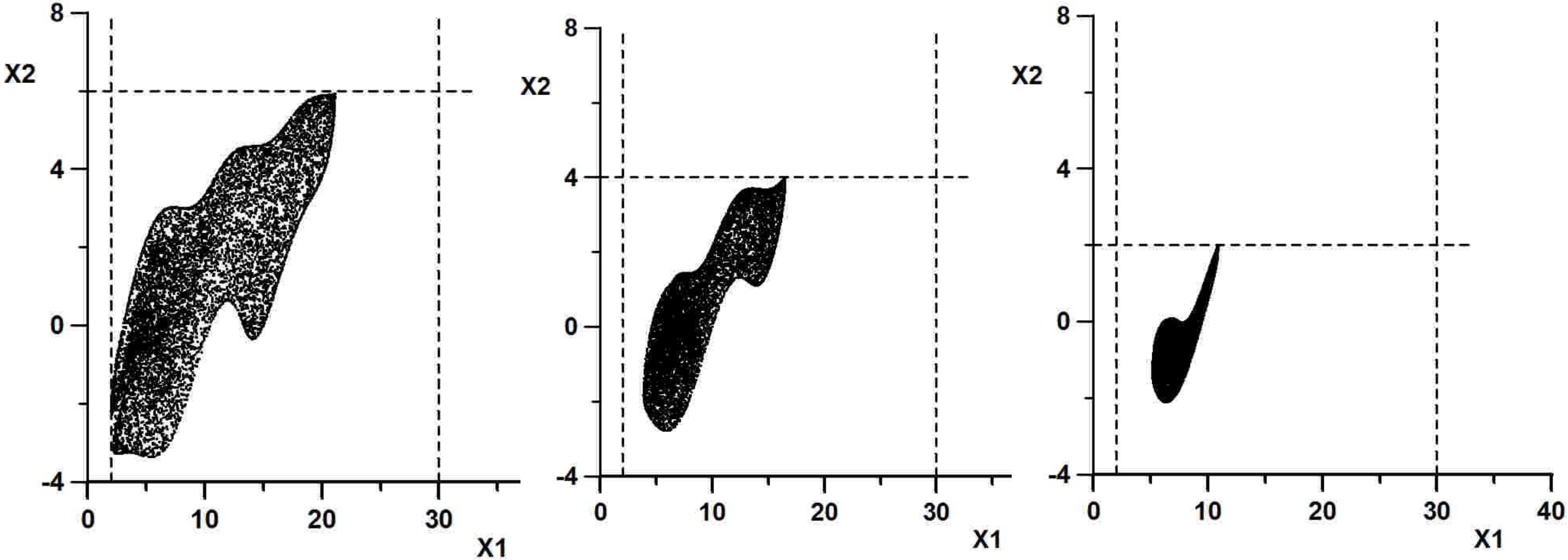
$$u^k(t) + \delta u^k(t), \left\| \delta u^k(t) \right\| \leq \Delta$$

# Test problem 02

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u_1 - \sin(x_1)$$

$$x_1(t_0) = 5,\ x_2(t_0) = 0,\quad |u(t)| \le \alpha,\ \ t \in [0,5]$$




| | | |
|---|---|---|
| $2 \le x_1(t_1) \le 30$ | $2 \le x_1(t_1) \le 30$ | $2 \le x_1(t_1) \le 30$ |
| $-4 \le x_2(t_1) \le 6$ | $-4 \le x_2(t_1) \le 4$ | $-4 \le x_2(t_1) \le 2$ |
| $\lvert u(t)\rvert \le \alpha^* = 1.145529$ | $\lvert u(t)\rvert \le \alpha^* = 0.783166$ | $\lvert u(t)\rvert \le \alpha^* = 0.406571$ |

# Popular approximative constructions

- “boxes”
- spheres
- ellipsoids
- meshes ("cloud")
- parallelotopes
- simplexes
- ovaloids
- …

# “Cloud approximation”

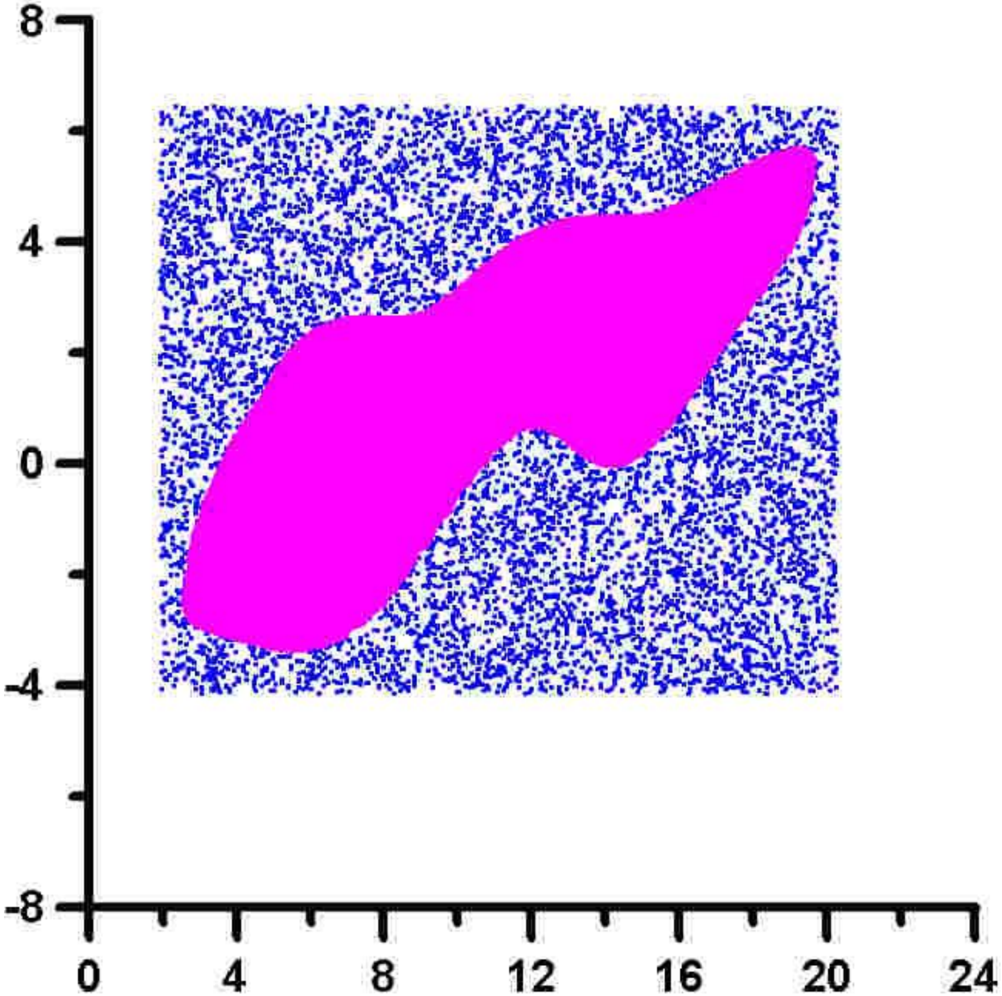
## Algorithms for realization of "set-theoretic" operations

- association
- intersection
- addition
- convexication
- delineation
- boundary approximation
- evaluation of the "diameter“
- estimation of "spread“
- estimate the volume of the set

# “Cloud approximation”
## Algorithm for addition constructing

- outlining the original "cloud" with a box
- generation of test points from a box
- fixation in the "cloud"-addition of test points far from the points of the original "cloud"

# “Cloud approximation”
## Algorithm for addition constructing



# “Cloud approximation”
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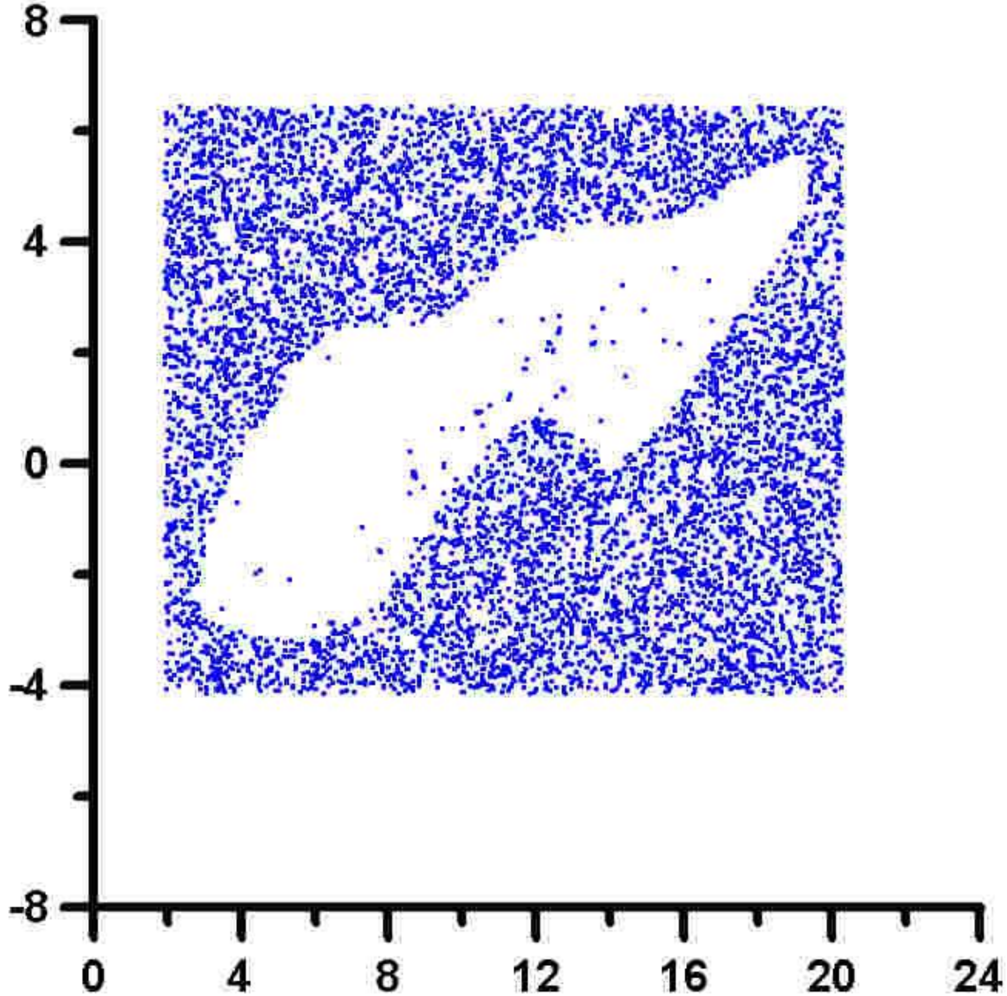
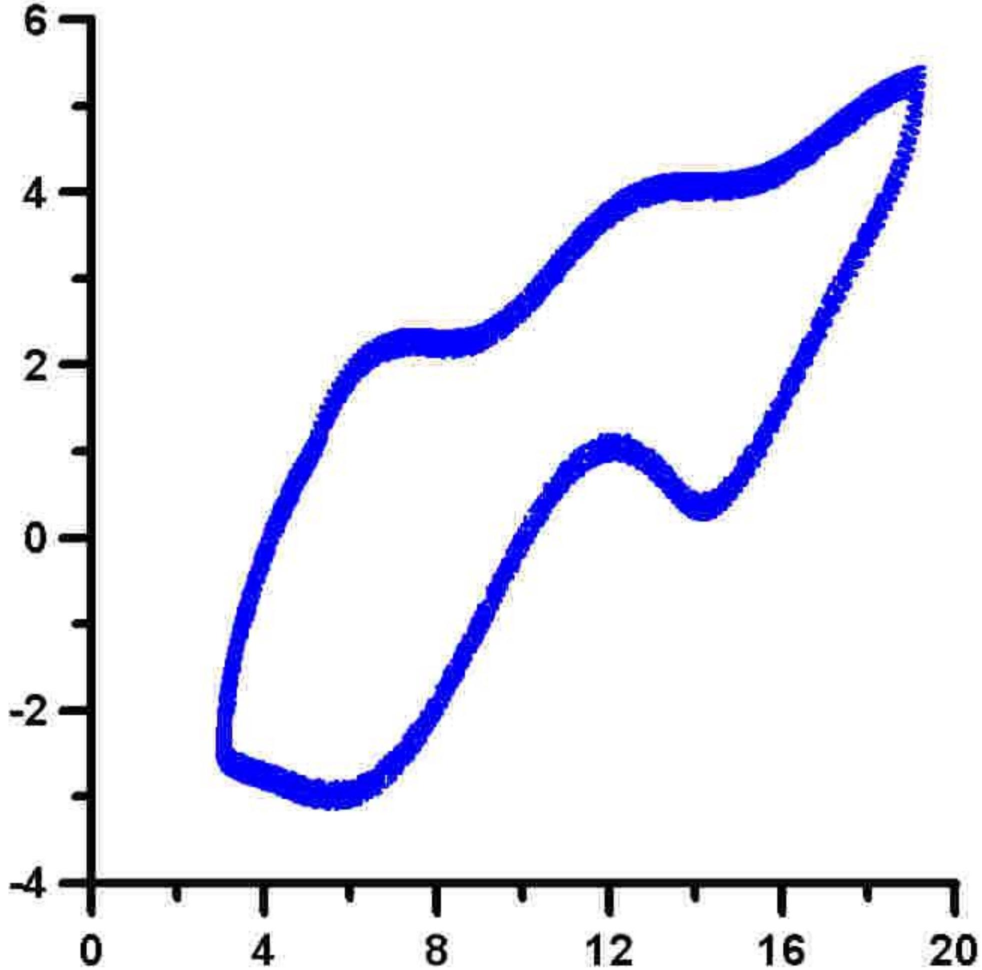


# “Cloud approximation”
## Algorithm for the boundary approximation

- the construction of a "cloud"-addition
- removal of points close to the points of the original "cloud"

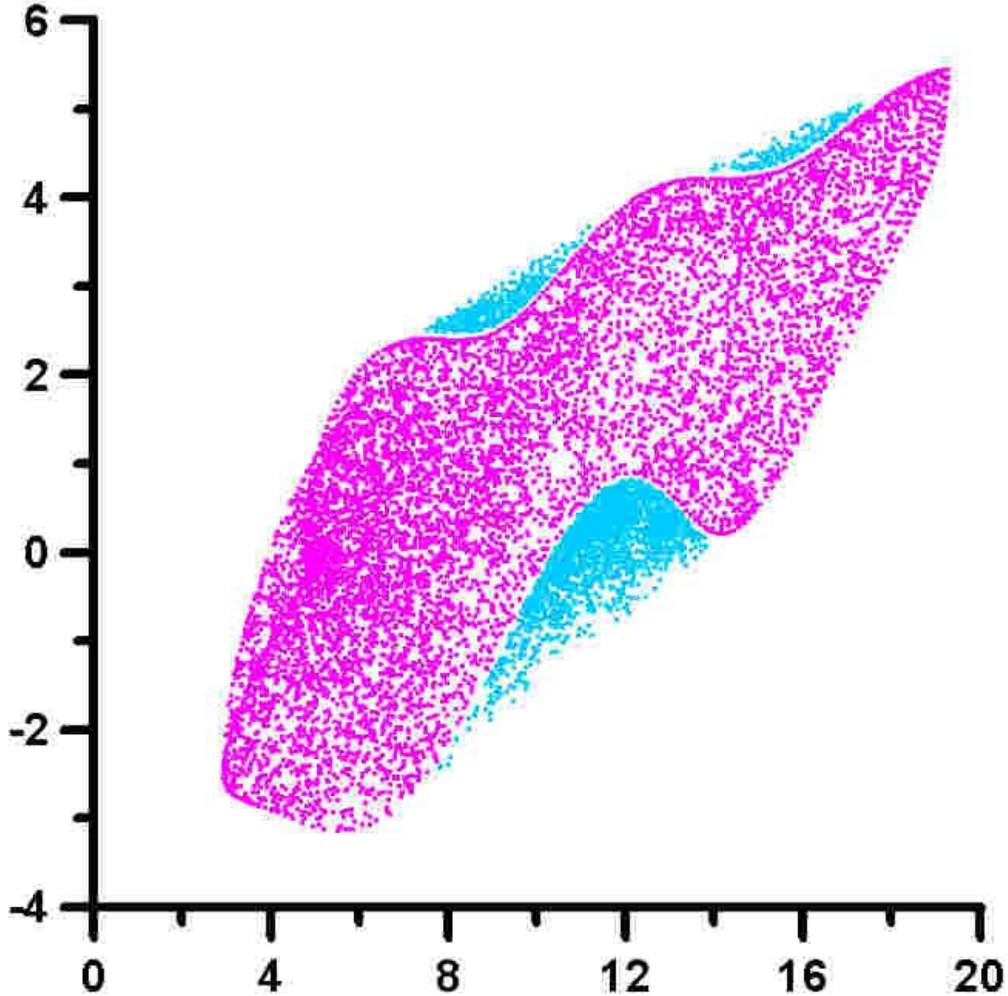
# “Cloud approximation”
## Algorithm for the boundary approximation



# “Cloud approximation”
## Algorithm for the convexification

- selecting pairs of points from the original “cloud“
- generation of test points on the connecting segments
- selection of test points, fixation of the points from the original “cloud” that do not fall in the its neighborhoods

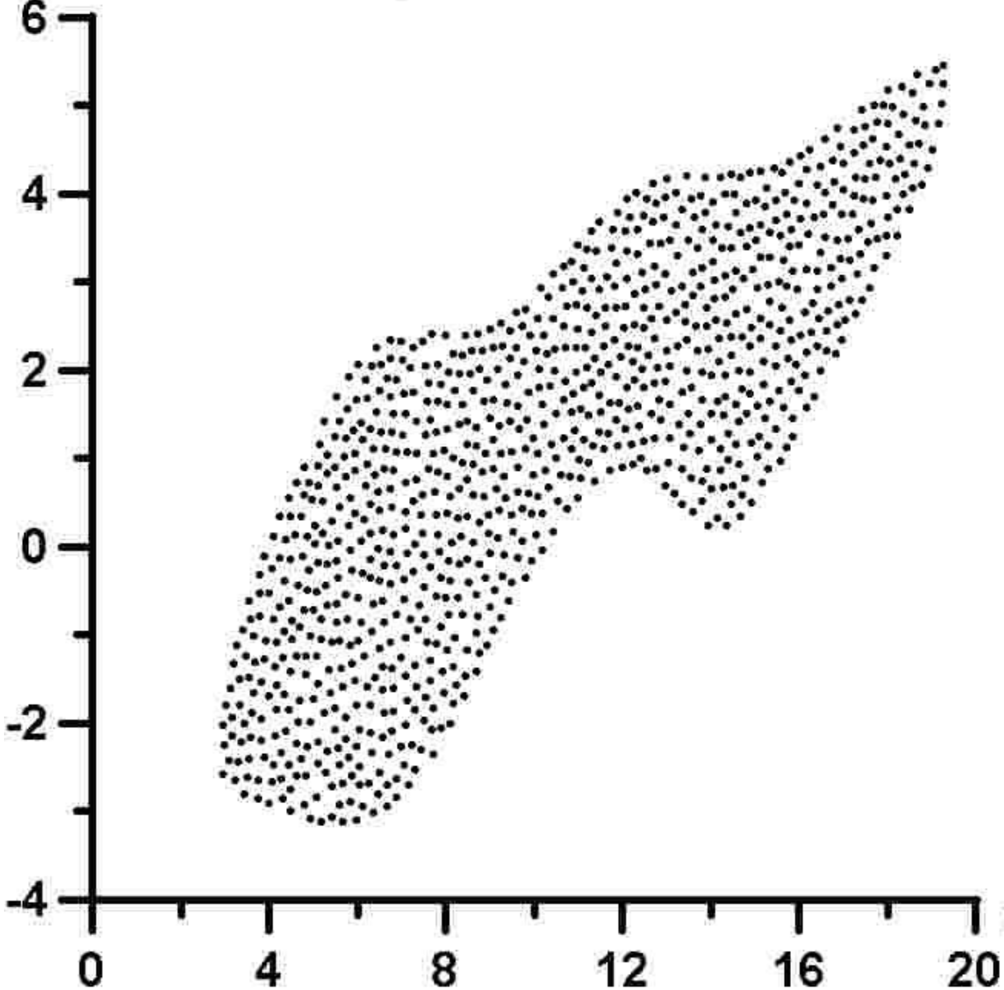
# “Cloud approximation”
## Algorithm for the convexification



# “Cloud approximation”
## Algorithm of quasiuniform filling

- generation a start "cloud" from one point
- generation of test points
- selection of test points, fixation of points not including in the neighborhood of a point of the already existing "cloud"

# “Cloud approximation”
## Algorithm of quasiuniform filling



# “Cloud approximation”
## The algorithm for estimating the volume of the set ("Archimedes' algorithm")

- outlining the "cloud" with the box
- quasiuniform filling of the "cloud“
- quasiuniform filling of the contouring box
- volume estimation through the ratio of the points number in the set and the box approximations

# Estimation of the cluster volume
## Archimedes' algorithm

- fixe $R$ – radius of the test sphere
- “the enclosing” box is constructed
- calculate $N$ – the number of cluster elements lying at least $R$ from each other
- calculate $M$ – the number of disjoint spheres of radius $R$ that fill the enclosing box
- cluster volume estimation is equal the box volume * $M$ / $N$

# The FOREL clustering method

## "Formal Element"

- convergence is proved in a finite number of steps
- strong dependence on the choice of the first point
- relatively low productivity
- **close to linear computational complexity**


1) Zagoruiko N.G., Yolkina V.N., Lbov G.S. Algorithms for detecting empirical regularities. – Novosibirsk: Science, 1985. – 999 p. (In Russian).
2) Zagoruiko N.G. Applied methods of data and knowledge analysis. – Novosibirsk: IM SB RAS, 1999. – 270 p. – ISBN 5-86134-060-9.

# Software OPTCON-SV (version 0.5)

- algorithms for estimating the record value of the function
- algorithms for generation of "cloud approximation“
- FOREL algorithm for clustering
- tools for research, fixation and visualization of the clusters

# Software
# OPTCON-SV
## Algorithms for "clouds“ generation

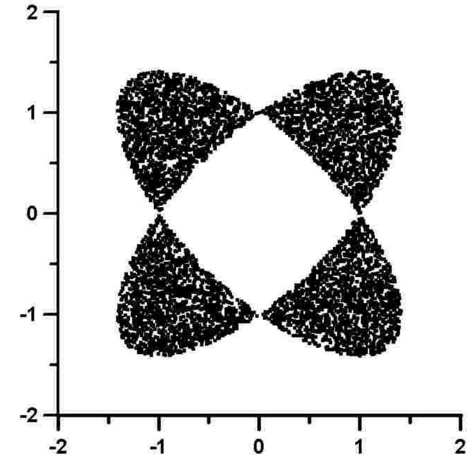
- stochastic approximation algorithm
- algorithm of approximation with a search "along the Polak“
- approximation algorithm with Hooke-Jeeves search
- algorithm of deterministic approximation for the function of two variables

# Software OPTCON-SV
## Tools

- table of distances between the centers of the clusters
- sphere chart of the cluster
- coordinate cluster chart
- cluster estimation algorithm of Archimedes
- the lower bound of the function value on the cluster

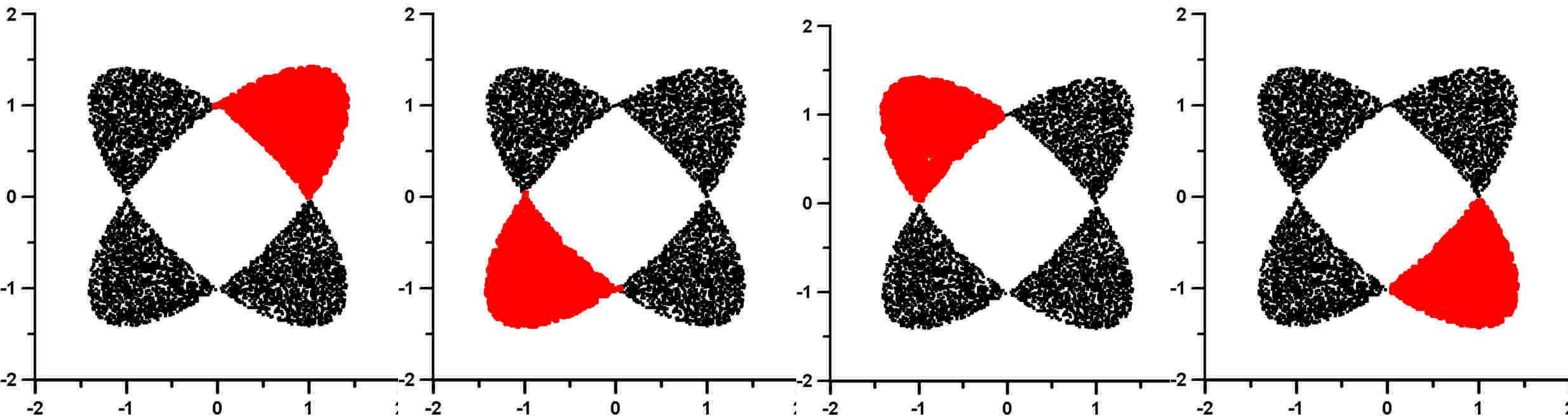
# Test problem 03

**calculation by the method of stochastic approximation, 1 min. of CPU time, about 600,000 samples, found 6714 points**




$$f(x_1, x_2) = (x_1 - 1)^2 (x_1 + 1)^2 + (x_2 - 1)^2 (x_2 + 1)^2$$

$$f(x_1, x_2) \le 1$$



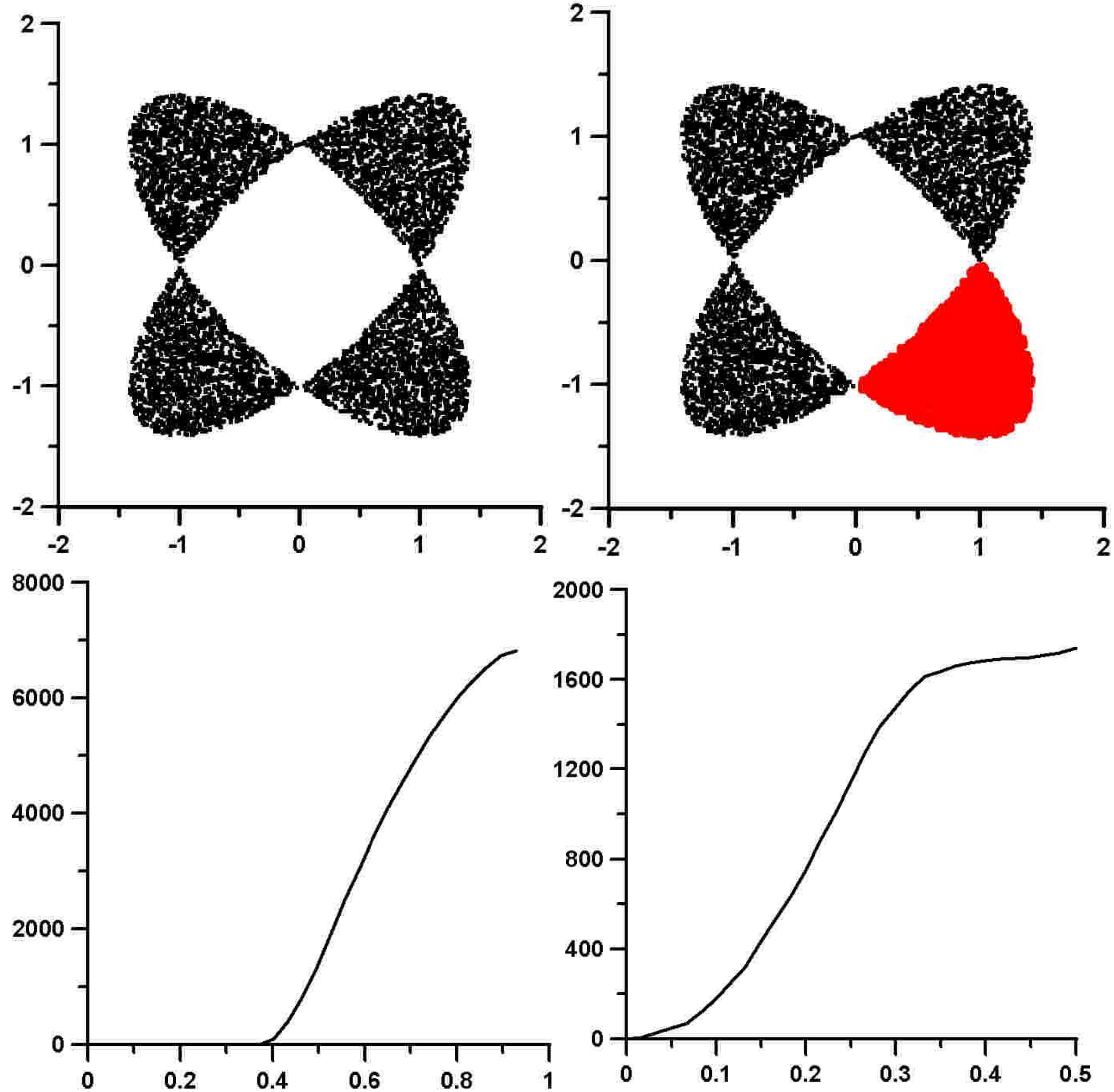
# “Cloud approximation”

## What do we want and what is obtained with the help of "prequadratic" algorithms

- find an estimate of the record value of the function
- find the estimate of the location region of the global extremum
- reduce the volume of the search box
- exclude unpromising areas
- increase the probability of finding a global extremum
- bypass the hard gullies

# Sphere chart of the cluster

**The number of cluster points falling into a sphere of increasing radius with center at the center of gravity of the cluster**




## Table of distances between cluster centers

| | 1 | 2 | 3 |
|---|---|---|---|
| 2 | 0.975 | | |
| 3 | 0.696 | 0.700 | |
| 4 | 0.695 | 0.701 | 1.000 |

# Conclusions

- simple algorithms give a good result
- it is possible to evaluate sets with complex geometry
- there is no strict limitation in dimension
- there is potential for parallelization

- ***"Cloud" approximations can be a useful tool***

# Thank you for attention!

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